Derivation of the Attenuation Rate Formula in Equation (18) from Woodhouse's Modal Damping Definition

The damping factors of a guitar string as a function of the mode number is given by Woodhouse in Ref. 21 (page 956) as

$$\eta_i = \frac{T\left(\eta_{\mathrm{F}} + \eta_{\mathrm{A}}/\omega_{0,i}\right) EI\eta_{\mathrm{B}}\beta_i^2}{T + EI\beta_i^2}, \tag{C.1}$$

where $\beta_i = i\pi/L$ and

$$\omega_{0,i} = \frac{EI\beta_i^4 + T\beta_i^2}{\rho A} \tag{C.2}$$

is the mode natural frequency[1]. The supscripts "A", "F", and "B" indicate that the factors in (C.1) represent damping due to "air", "friction", and "bending", respectively. Dividing both numerator and denominator in (C.1) by the tension T, one obtains

$$\eta_i = \frac{\eta_{\mathrm{A}}/\omega_0 + \eta_{\mathrm{F}} + \lambda\eta_{\mathrm{B}}\beta_i^2}{1 + \lambda\beta_i^2}, \tag{C.3}$$

where the ratio $\lambda = EI/T$ is a useful indicator of the stiffness-like character of the string, which relates to the inharmonicity factor as $\mathcal{B} = \lambda(\pi/L)^2$. The modal attenuation rate is related to the damping factor as

$$\begin{aligned} \alpha_i &= \frac{1}{2}\omega_{0,i}\eta_i \\ &= \frac{\eta_{\mathrm{A}} + \left(\eta_{\mathrm{F}} + \lambda\eta_{\mathrm{B}}\beta_i^2\right)\omega_{0,i}}{2\left(1 + \lambda\beta_i^2\right)}, \end{aligned} \tag{C.4}$$

which, after substitution of (C.2), can be written

$$\begin{aligned} \alpha_i &= \frac{\eta_{\mathrm{A}} + \sqrt{\frac{T}{\rho A}}\left(\eta_{\mathrm{F}} + \lambda\eta_{\mathrm{B}}\beta_i^2\right)\sqrt{1 + \lambda\beta_i^2}}{2\left(1 + \lambda\beta_i^2\right)} \\ &= \frac{\sigma_0 + \left(\sigma_1 + \sigma_3\beta_i^2\right)\sqrt{1 + \lambda\beta_i^2}}{\left(1 + \lambda\beta_i^2\right)}, \end{aligned} \tag{C.5}$$

where, with $c = \sqrt{\frac{T}{\rho A}}$ representing the wave velocity in the absence of stiffness, the coefficients are

$$\sigma_0 = \frac{1}{2}\eta_{\mathrm{A}}, \quad \sigma_1 = \frac{1}{2}c\eta_{\mathrm{F}}, \quad \sigma_3 = \frac{1}{2}\lambda c\eta_{\mathrm{B}}. \tag{C.6}$$

Since musical strings are generally tension-dominated, i.e. $\lambda\beta_i^2 \ll 1$ in the audio frequency range, we may simplify (C.5) to

$$\alpha_i \approx \sigma_0 + \sigma_1\beta_i + \sigma_3\beta_i^3. \tag{C.7}$$

Hence the relationships in (C.6) approximately hold when using (C.7), which matches equation (18) in the paper.

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[1] Note this differs slightly from the actual modal frequency $\omega_i = \sqrt{\omega_{0,i}^2 - \alpha_i^2}$.