Step-By-Step Derivation of the Analytic Solution of Equation (50)

The nonlinear equation to be solved is

$$s_{\mathrm{b}} + u + \theta \frac{V_{\mathrm{b}}(\Delta y^n + s_{\mathrm{b}}) - V_{\mathrm{b}}(\Delta y^n)}{s_{\mathrm{b}}} = 0. \tag{A.1}$$

Substituting equation (30) of the paper yields

$$s_{\mathrm{b}} + u + \varphi \frac{\lfloor \Delta y^n - s_{\mathrm{b}} \rfloor^2 - \lfloor \Delta y^n \rfloor^2}{s_{\mathrm{b}}} = 0. \tag{A.2}$$

where $\Delta y^n = h_{\mathrm{b}} - y_{\mathrm{b}}^n$ is the 'compression' at the previous time step (n) and $\varphi = \frac{1}{2}\theta k_{\mathrm{b}}$.

No Contact at the Previous Step

For the case that there was no contact at the previous time step, i.e.

$$\Delta y^n \leq 0, \tag{A.3}$$

we have that if

$$s_b \geq \Delta y^n, \tag{A.4}$$

equation (A.2) reduces to

$$s_{\mathrm{b}} + u = 0. \tag{A.5}$$

Thus in that case $s_{\mathrm{b}} = -u$ and therefore, from (A.4) it follows that then

$$u \leq -\Delta y^n. \tag{A.6}$$

On the other hand if

$$s_{\mathrm{b}} < \Delta y^n, \tag{A.7}$$

the nonlinear equation reduces to

$$s_{\mathrm{b}} + u + \varphi \frac{(\Delta y^n - s_{\mathrm{b}})^2}{s_{\mathrm{b}}} = 0 \tag{A.8}$$

or

$$(1 + \varphi)\,(s_{\mathrm{b}})^2 + (u - 2\varphi \Delta y^n)\, s_{\mathrm{b}} + \varphi (\Delta y^n)^2 = 0, \tag{A.9}$$

which has the solutions

$$s_{\mathrm{b}} = \frac{2\psi \Delta y^n - u \pm \sqrt{u^2 - 4\varphi \Delta y^n (u + \Delta y^n)}}{2(1+\varphi)}. \tag{A.10}$$

The term inside the square root is always positive since Δy^n is negative and $(u + \Delta y^n)$ is positive. Since we that $s_{\mathrm{b}} \leq \Delta y^n < 0$, the solution must be negative, which is assured only for the solution with the negative root, therefore

$$s_{\mathrm{b}} = \frac{2\psi \Delta y^n - u - \sqrt{u^2 - 4\varphi \Delta y^n (u + \Delta y^n)}}{2(1+\varphi)}. \tag{A.11}$$

Now, from (A.7) it follows that for this case

$$\Delta y^n > \frac{2\psi\Delta y^n - u - \sqrt{u^2 - 4\varphi\Delta y^n(u + \Delta y^n)}}{2(1+\varphi)}, \tag{A.12}$$

which can be written as

$$(2\Delta y^n + u)^2 > u^2 - 4\varphi\Delta y^n(u + \Delta y^n) \tag{A.13}$$

or

$$\Delta y\,(\Delta y + u)\,(1+\varphi) > 0. \tag{A.14}$$

Since $\Delta^n < 0$ and $\varphi > 0$, it follows that $(\Delta y^n + u) < 0$ thus

$$u > -\Delta y^n, \tag{A.15}$$

which is complementary to (A.6). Therefore testing (A.6) can serve as a computable criterion for deciding which of the two cases [(A.4) or (A.7)] is applicable.

Contact in the Previous Step

Now we assume

$$\Delta y^n > 0. \tag{A.16}$$

If

$$s_\mathrm{b} \le \Delta y^n, \tag{A.17}$$

then the nonlinear equaton reduces to

$$(1+\varphi)\,s_\mathrm{b} + u - 2\varphi\Delta y^n = 0. \tag{A.18}$$

therefore the solution then is

$$s_\mathrm{b} = \frac{u - 2\varphi\Delta y^n}{1+\varphi}. \tag{A.19}$$

and, from (A.17), we have

$$\Delta y^n \le \frac{u - 2\varphi\Delta y^n}{1+\varphi}. \tag{A.20}$$

Therefore the corresponding criterion can be expressed as

$$u \ge \Delta y^n\,(\varphi - 1)\,. \tag{A.21}$$

On the other hand if

$$s_\mathrm{b} > \Delta y^n, \tag{A.22}$$

the nonlinear equation reduces to

$$(s_\mathrm{b})^2 + u s_\mathrm{b} + \varphi(\Delta y^n)^2 = 0, \tag{A.23}$$

which has the solutions

$$s_\mathrm{b} = -\frac{1}{2}u \pm \frac{1}{2}\sqrt{u^2 + 4\varphi(\Delta y^n)^2}. \tag{A.24}$$

Given that $s_{\mathrm{b}} > \Delta y^n > 0$, s_{b} must be positive, which is assured only for the solution with the positive root. Hence the solution for this case is

$$s_{\mathrm{b}} = -\frac{1}{2}u + \frac{1}{2}\sqrt{u^2 + 4\varphi(\Delta y^n)^2}. \tag{A.25}$$

Therefore, from (A.22), we have that

$$2\Delta y^n < -u + \sqrt{u^2 + 4\varphi(\Delta y^n)^2} \tag{A.26}$$

which equates to

$$u^2 + 4\varphi(\Delta y^n)^2 > (2\Delta y^n + u)^2 \tag{A.27}$$

yielding the criterion

$$u < \Delta y^n\,(\varphi - 1)\,, \tag{A.28}$$

which is complementary to (A.21). Therefore testing (A.21) can serve as a computable criterion for deciding which of the two cases [(A.17) or (A.22)] is applicable.

Summary

Collecting all the cases, we have

$$\Delta y^n \leq 0 \quad \text{and} \quad u < -\Delta y^n : s_{\mathrm{b}} = -u \tag{A.29}$$

$$\Delta y^n \leq 0 \quad \text{and} \quad u \geq -\Delta y^n : s_{\mathrm{b}} = \frac{2\psi\Delta y^n - u - \sqrt{u^2 - 4\varphi\Delta y^n(u + \Delta y^n)}}{2(1+\varphi)} \tag{A.30}$$

$$\Delta y^n > 0 \quad \text{and} \quad u \geq \Delta y^n(\varphi - 1) : s_{\mathrm{b}} = \frac{u - 2\varphi\Delta y^n}{1+\varphi} \tag{A.31}$$

$$\Delta y^n > 0 \quad \text{and} \quad u < \Delta y^n(\varphi - 1) : s_{\mathrm{b}} = -\frac{1}{2}u + \frac{1}{2}\sqrt{u^2 + 4\varphi(\Delta y^n)^2} \tag{A.32}$$